Extra Problems - 2/14

① Show that the function $f(\vec{x}) = |\vec{x}|$ is continuous on $\mathbb{R}^n$. <u>Hint</u>: $|\vec{x}-\vec{a}|^2 = (\vec{x}-\vec{a})\cdot(\vec{x}-\vec{a})$

<u>Solution</u>

Recall that f is continuous at $\vec{a}$ if

$$\lim_{\vec{x}\to\vec{a}} f(\vec{x}) = f(\vec{a})$$

and is continuous on $\mathbb{R}^n$ if it is continuous at all $\vec{a}\in\mathbb{R}^n$.

Choose any $\vec{a}\in\mathbb{R}^n$, and let $\varepsilon>0$. Choose $\delta=\varepsilon$.

By the hint:

$$|\vec{x}-\vec{a}|^2 = |\vec{x}|^2+|\vec{a}|^2-2\vec{x}\cdot\vec{a} = |\vec{x}|^2+|\vec{a}|^2-2|\vec{x}||\vec{a}|\cos\theta$$

$$\geq |\vec{x}|^2+|\vec{a}|^2-2|\vec{x}||\vec{a}| = (|\vec{x}|-|\vec{a}|)^2$$

$$\Rightarrow |\vec{x}-\vec{a}| \geq \big||\vec{x}|-|\vec{a}|\big|$$

Then, if $0<|\vec{x}-\vec{a}|<\delta$, we have

$$\varepsilon = \delta > |\vec{x}-\vec{a}| \geq \big||\vec{x}|-|\vec{a}|\big| = |f(\vec{x})-f(\vec{a})|$$

Thus $\lim_{\vec{x}\to\vec{a}} f(\vec{x}) = f(\vec{a})$.

Since this is true for any $\vec{a}\in\mathbb{R}^n$ (we used an arbitrary $\vec{a}$ above), f is continuous on $\mathbb{R}^n$.

② Let

$$f(x,y)=\begin{cases}\dfrac{x^3y-xy^3}{x^2+y^2} & \text{if } (x,y)\neq(0,0)\\ 0 & \text{if } (x,y)=(0,0)\end{cases}$$

a) Find $f_x(x,y)$ and $f_y(x,y)$ when $(x,y)\neq(0,0)$

b) Find $f_x(0,0)$ and $f_y(0,0)$.

c) Find $f_{xy}(0,0)$ and $f_{yx}(0,0)$.

d) Explain your answer in part c. Is there a contradiction?

Solution

a) $$f_x(x,y)=\frac{(3x^2y-y^3)(x^2+y^2)-(x^3y-xy^3)(2x)}{(x^2+y^2)^2}=\frac{3x^4y-x^2y^3+3x^2y^3-y^5-2x^4y+2x^2y^3}{(x^2+y^2)^2}$$

$$=\frac{x^4y+4x^2y^3-y^5}{(x^2+y^2)^2}$$

By symmetry $$f_y(x,y)=\frac{-xy^4+4x^3y^2+x^5}{(x^2+y^2)^2}$$

b) Need to use the definition here since f is piecewise defined.

$$f_x(0,0)=\lim_{h\to0}\frac{f(0+h,0)-f(0,0)}{h}=\lim_{h\to0}\frac{\frac{0}{h^2}-0}{h}=\lim_{h\to0}0=0$$

$$f_y(0,0)=\lim_{h\to0}\frac{f(0,0+h)-f(0,0)}{h}=\lim_{h\to0}\frac{\frac{0}{h^2}-0}{h}=\lim_{h\to0}0=0$$

c) $$f_{xy}(0,0)=\lim_{h\to0}\frac{f_x(0,0+h)-f_x(0,0)}{h}=\lim_{h\to0}\frac{\frac{-h^5}{h^4}-0}{h}=\lim_{h\to0}\frac{-h}{h}=-1$$

$$f_{yx}(0,0)=\lim_{h\to0}\frac{f_y(0+h,0)-f_y(0,0)}{h}=\lim_{h\to0}\frac{\frac{h^5}{h^4}-0}{h}=\lim_{h\to0}\frac{h}{h}=1$$

d) So, we have $f_{xy}(0,0) \neq f_{yx}(0,0)$. Remember that, for Clairaut's theorem, both f_{xy} and f_{yx} have to be continuous in order to have $f_{xy} = f_{yx}$. Away from $(0,0)$ (i.e., $(x,y) \neq (0,0)$) we have

$$f_{xy} = \frac{x^6 + 9x^4y^2 - 9x^2y^4 - y^6}{(x^2+y^2)^3}$$

Taking the limit along the line $y = mx$, we have

$$\lim_{(x,mx)\to(0,0)} f_{xy}(x,y) = \lim_{x\to 0} \frac{x^6 + 9m^2x^6 - 9m^4x^6 - m^6x^6}{(x^2+m^2x^2)^3}$$

$$= \lim_{x\to 0} \frac{\cancel{x^6}(1+9m^2-9m^4-m^6)}{\cancel{x^6}(1+m^2)^3}$$

$$= \frac{1+9m^2-9m^4-m^6}{(1+m^2)^3}$$

which depends on m. Thus the limit at $(0,0)$ of f_{xy} does not exist, hence f_{xy} is not continuous at $(0,0)$ and so Clairaut's theorem does not apply in this case. So, no, there is no contradiction

③ Assume f and g are C^2 functions. Show that any function of the form

$$z = f(x+at) + g(x-at)$$

is a solution of the wave equation

$$\frac{\partial^2 z}{\partial t^2} = a^2 \frac{\partial^2 z}{\partial x^2}$$

<u>Hint</u>: Let $u = x+at$, $v = x-at$.

<u>Solution</u>: By the hint, we have $z = f(u)+g(v)$.

Then: $$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial t} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial t} = f'(u)(a) + g'(v)(-a) = af'(u) - ag'(v)$$
$$= a\left(f'(x+at) - g'(x-at)\right) = z_t$$
$$\frac{\partial^2 z}{\partial t^2} = \frac{\partial z_t}{\partial u}\frac{\partial u}{\partial t} + \frac{\partial z_t}{\partial v}\frac{\partial v}{\partial t} = a\left(f''(u)(a) - g''(v)(-a)\right)$$
$$= a^2\left(f''(u) + g''(v)\right) = a^2\left(f''(x+at) + g''(x-at)\right)$$
$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x} = f'(u)\cdot(1) + g'(v)(1) = f'(u) + g'(v)$$
$$= f'(x+at) + g'(x-at) = z_x$$
$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial z_x}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z_x}{\partial v}\frac{\partial v}{\partial x} = f''(u)(1) + g''(v)(1)$$
$$= f''(x+at) + g''(x-at)$$

Plug these in:

$$a^2\left(f''(x+at) + g''(x-at)\right) = \frac{\partial^2 z}{\partial t^2} \overset{?}{=} a^2 \frac{\partial^2 z}{\partial x^2} = a^2\left(f''(x+at) + g''(x-at)\right)$$

So the equality checks!